

Presymplectic Geometry and Fermat's Principle for anisotropic media

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ABSTRACT

The tools of presymplectic geometry are used to study light rays trajectories in anisotropic media.

The usefulness of the Lie–Hamilton optics in many different problems as ray tracing for ray design and computation of aberrations suggests the study of what happens for anisotropic media, because of the recent interest in the use of anisotropic optical material. This motivated a very recent paper [1] where the Hamiltonian formulation of geometric anisotropic optics was studied. The theory was reexamined in [2]. The basic principle of the theory is the celebrated Fermat's principle of least time (or extremal time if reflection is also allowed). In other words, light rays connecting points A and B are lines in the space in such a way that they satisfy the following variational condition:

$$\delta \int_A^B n \, ds = 0. \quad (1)$$

The refractive index of the medium is given by the quotient $n = \frac{c}{v}$, and then physics tell us that $n > 1$.

In a recent paper [3] we analysed from a geometric perspective the relationship for the case of isotropic media of the problem of determination of extremal curves for (1) with that of geodesics of a Riemannian metric conformal to the Euclidean metric. We will consider here the case in which the medium is not isotropic but it may depend on the velocity, or more specifically, on the direction of the ray. In this last case the problem cannot be reduced to a problem of Riemannian geometry as it happened when the refractive index n only depended on the position. So, the techniques of Presymplectic geometry are unavoidable for dealing with this dependence of the refractive index with the ray direction.

To begin we remark the strong similarity of Fermat's principle with the more traditional Hamilton's principle of Classical Mechanics, with a Lagrangian function given by

$$L = n \sqrt{g(v, v)}. \quad (2)$$

This Lagrangian function is differentiable only in the set of velocity phase space obtained by removing the null velocity points, i.e., the zero section of the tangent bundle. Moreover, the Lagrangian L is homogeneous of degree one,

$$v^i \frac{\partial L}{\partial v^i} = L, \quad (3)$$

and consequently the corresponding energy function vanishes identically. Therefore the Lagrangian is singular, because taking derivatives with respect to v^j of both sides of the preceding equation we obtain

$$\frac{\partial^2 L}{\partial v^i \partial v^j} v^i = 0, \quad (4)$$

and then the Hessian matrix

$$W_{ij} = \frac{\partial^2 L}{\partial v^i \partial v^j} \quad (5)$$

is singular. The theory should be carefully reexamined using the tools of Presymplectic Geometry, as it was done in [3] for the isotropic media, where the tools of Riemannian Geometry were shown to be very useful. Actually, it is possible to show that in the latter case the solution curves for the regular Lagrangian system described by the regular Lagrangian

$$\mathbb{L} = \frac{1}{2} n^2 g(v, v) \quad (6)$$

are just the curves solution of the original problem, even if the curves are reparametrized (see e.g. [4] and references therein). Our aim is to analyse what happens in the more general case in which the refractive index can depend on the ray direction, i.e., the refractive index is a homogeneous function of degree zero of velocities in the set obtained by removing the zero section of the velocity phase space.

The geometric approach to Lagrangian Classical mechanics uses as velocity phase space the tangent bundle TM of the configuration space M that is assumed to be a differentiable manifold of dimension N . From now on we will follow the notation used in [5]. The tangent structure is characterized by a $(1, 1)$ -tensor field called vertical endomorphism S that in terms of natural coordinates (q^i, v^i) of the tangent bundle TM is given by

$$S = \frac{\partial}{\partial v^i} \otimes dq^i. \quad (7)$$

Given a function $L \in C^\infty(TM)$, we may define an exact 2-form in TM , $\omega_L = -d\theta_L$, with the 1-form θ_L being defined by $\theta_L = dL \circ S$, and a function $E_L = \Delta(L) - L$, called energy function. In the above mentioned coordinates of TM we have the following expressions:

$$\theta_L = \frac{\partial L}{\partial v^i} dq^i, \quad (8)$$

$$\omega_L = \frac{\partial^2 L}{\partial q^i \partial v^j} dq^j \wedge dq^i + \frac{\partial^2 L}{\partial v^i \partial v^j} dq^i \wedge dv^j, \quad (9)$$

$$\Delta = v^i \frac{\partial}{\partial v^i} \quad (10)$$

$$E_L = v^i \frac{\partial L}{\partial v^i} - L, \quad (11)$$

Here $\Delta \in \mathfrak{X}(TM)$ denotes the Liouville vector field generating dilations along the fibres. When ω_L is nondegenerate, i.e., the Hessian matrix (5) is regular, it defines a symplectic structure on TM , and a vector field Γ_L uniquely determined by $i(\Gamma_L)\omega_L = dE_L$.

We will next examine the problem of light rays, even for the more general case in which the refractive index n depends on the ray direction. This means that the refractive index must be homogeneous of degree zero in the velocities, $\Delta n = 0$. We define a new Lagrangian $\mathbb{L} = \frac{1}{2}L^2$. The Lagrangian \mathbb{L} is homogeneous of degree two in the velocities, $\Delta\mathbb{L} = 2\mathbb{L} = L^2$, and then $E_{\mathbb{L}} = \mathbb{L}$.

Now taking into account that $\theta_{\mathbb{L}} = d\mathbb{L} \circ S$ we see that the Liouville 1-form $\theta_{\mathbb{L}}$ is proportional to θ_L , namely, $\theta_{\mathbb{L}} = L\theta_L$, (see [6]) and then

$$\omega_L = \frac{1}{L}\omega_{\mathbb{L}} + \frac{1}{L^3}d\mathbb{L} \wedge \theta_{\mathbb{L}}. \quad (12)$$

As indicated above, when the refractive index n does not depend on velocities the 2-form $\omega_{\mathbb{L}}$ is regular. In this more general case, however, it may be singular, because

$$\omega_{\mathbb{L}} = L\omega_L - dL \wedge \theta_L \quad (13)$$

means that

$$\omega_{\mathbb{L}}^{\wedge N} = L^N \omega_L^{\wedge N} - N L^{N-1} dL \wedge \theta_L \wedge \omega_L^{\wedge(N-1)} = -N L^{N-1} dL \wedge \theta_L \wedge \omega_L^{\wedge(N-1)}, \quad (14)$$

and then $\omega_{\mathbb{L}}^{\wedge N}$ can be identically null and in this case \mathbb{L} would be singular.

In the following we will restrict ourselves to the case in which \mathbb{L} is regular and therefore there will be a uniquely defined vector field $\Gamma_{\mathbb{L}}$ such that

$$i(\Gamma_{\mathbb{L}})\omega_{\mathbb{L}} = dE_{\mathbb{L}} = d\mathbb{L}, \quad (15)$$

and then

$$i(\Gamma_{\mathbb{L}})(d\mathbb{L}) = \Gamma_{\mathbb{L}}(\mathbb{L}) = 0. \quad (16)$$

Moreover, $\Gamma_{\mathbb{L}}$ is a second order differential equation vector field.

First we check that the Liouville vector field Δ lies in $\ker \omega_L$. Indeed,

$$i(\Delta)\omega_L = \frac{1}{L}i(\Delta)\omega_{\mathbb{L}} + \frac{1}{L^3}\Delta\mathbb{L}\theta_{\mathbb{L}} - \frac{1}{L^3}d\mathbb{L}(i(\Delta)\theta_{\mathbb{L}}), \quad (17)$$

and taking into account that Δ is vertical and $\theta_{\mathbb{L}}$ semibasic, the last term vanishes. Moreover,

$$i(\Delta)\omega_{\mathbb{L}} = -\mathcal{L}_{\Delta}\theta_{\mathbb{L}} + d(i(\Delta)\theta_{\mathbb{L}}) = -\mathcal{L}_{\Delta}\theta_{\mathbb{L}} = -\theta_{\mathbb{L}} \quad (18)$$

and therefore, taking into account that $\Delta\mathbb{L} = 2\mathbb{L} = L^2$ we find that

$$i(\Delta)\omega_L = 0. \quad (19)$$

Secondly, $\Gamma_{\mathbb{L}}$ is also in the kernel of ω_L , because

$$i(\Gamma_{\mathbb{L}})\omega_L = \frac{1}{L}i(\Gamma_{\mathbb{L}})\omega_{\mathbb{L}} + \frac{1}{L^3}(\Gamma_{\mathbb{L}}\mathbb{L})\theta_{\mathbb{L}} - \frac{1}{L^3}d\mathbb{L}[i(\Gamma_{\mathbb{L}})\theta_{\mathbb{L}}] \quad (20)$$

and Γ_L being a SODE,

$$i(\Gamma_{\mathbb{L}})\theta_{\mathbb{L}} = (d\mathbb{L} \circ S)(\Gamma_{\mathbb{L}}) = \Delta\mathbb{L} = 2\mathbb{L} = L^2 \quad (21)$$

and therefore

$$i(\Gamma_{\mathbb{L}})\omega_L = 0. \quad (22)$$

Finally under the assumption that \mathbb{L} is regular, $\ker \omega_L$ is generated by Δ and $\Gamma_{\mathbb{L}}$. Indeed, given a vertical vector field $V \in \ker \omega_L$, then,

$$0 = i(V)\omega_L = \frac{1}{L}i(V)\omega_{\mathbb{L}} + \frac{1}{L^3}V(\mathbb{L})\theta_{\mathbb{L}} \quad (23)$$

and in particular

$$\frac{1}{L}i(\Delta)\omega_{\mathbb{L}} = -\frac{1}{L^3}\Delta(\mathbb{L})\theta_{\mathbb{L}} = -\frac{1}{L}\theta_{\mathbb{L}} \quad (24)$$

and therefore

$$i(V)\omega_{\mathbb{L}} = \frac{V(\mathbb{L})}{L^2}i(\Delta)\omega_{\mathbb{L}} \quad (25)$$

and as $\omega_{\mathbb{L}}$ is assumed to be regular, V should be proportional to Δ . Then, $\dim V(\ker \omega_L) = 1$ and because of the relation $\dim(\ker \omega_L) \leq 2 \dim V(\ker \omega_L)$ (see [7]), we can conclude that $\dim(\ker \omega_L) = 2$.

Under these circumstances is possible to apply the reduction theory of presymplectic manifolds, following the ideas developed by Marsden and Weinstein [8]. Presymplectic structures may arise either when using some constants of motion for reducing the phase space or also when the Lagrangian that has been chosen is singular. Then we will have a pair (P_0, Ω_0) where Ω_0 is a closed but degenerate 2-form. A consistent solution of the dynamical equation can only be found in some points, leading in this way to the final constraint submanifold P introduced by Dirac (see e.g. [9]). The pull back Ω of the form Ω_0 on this manifold will be assumed to be of constant rank. The recipe for dealing with these systems was given by Marsden and Weinstein [8]. First, in every point $m \in P$, $\ker \Omega_m$ is a k -dimensional linear space, so defining what is called a k -dimensional distribution. The important point is that closedness of Ω is enough to warrant that the distribution is integrable (and then it is called foliation): for any point $m \in P$, there is a k -dimensional submanifold of P passing through m and such that the tangent space at any point m' of this surface coincides with $\ker \Omega_{m'}$. Such integral k -dimensional submanifolds give a foliation of P by disjoint leaves and in the case in which the quotient space $\tilde{P} = P / \ker \Omega$ is a differentiable manifold, then it is possible to define a nondegenerate closed 2-form $\tilde{\Omega}$ in \tilde{P} such that $\tilde{\pi}^*\tilde{\Omega} = \Omega$. Here $\tilde{\pi} : P \rightarrow \tilde{P}$ is the natural projection. It suffices to define $\tilde{\Omega}(\tilde{v}_1, \tilde{v}_2) = \Omega(v_1, v_2)$, where v_1 and v_2 are tangent vectors to P projecting under $\tilde{\pi}_*$ onto \tilde{v}_1 and \tilde{v}_2 respectively. The symplectic space $(\tilde{P}, \tilde{\Omega})$ is said to be the reduced space. We will illustrate the method finding coordinates adapted to the distribution defined by the kernel $\ker \omega_L$ of the presymplectic structure defined by the singular optical Lagrangian in the case of a system in which either the index n depends on the third coordinate x^3 alone or the very interesting case in which the system is anisotropic and n is a function of the ray direction. We will determine the quotient reduced space and we will look for Darboux coordinates in this reduced symplectic manifold. Once Darboux coordinates have been found we can consider the problem from the active viewpoint and take advantage of the algebraic methods recently developed for computing aberrations (see e.g. [10]).

Let us now consider the most general isotropic case in which the refractive index of the medium is not constant but it is given by a smooth function $n(x^1, x^2, x^3)$. Fermat's principle suggests us to consider the corresponding mechanical problem described by a singular Lagrangian $L(q, v) = [g(v, v)]^{1/2}$, where g is a metric conformal to the Euclidean metric g_0 ,

$$g(v, w) = n^2 g_0(v, w). \quad (26)$$

This problem was analysed in [6] where, as above is cited, it was shown that its study can be reduced to that of a regular Lagrangian $\mathbb{L} = \frac{1}{2}L^2$. This Lagrangian \mathbb{L} is quadratic in velocities and the dynamical vector field $\Gamma_{\mathbb{L}}$ solution of the dynamical equation $i(\Gamma_{\mathbb{L}})\omega_{\mathbb{L}} = dE_{\mathbb{L}} = d\mathbb{L}$ is not only a second order differential equation vector field but, moreover, it is a spray [11], the projection onto \mathbb{R}^3 of its integral curves being the geodesics of the Levi-Civita connection defined by g . Then, $\Gamma_{\mathbb{L}}$ is the geodesic spray given by

$$\Gamma_{\mathbb{L}} = v^i \frac{\partial}{\partial q^i} - \Gamma^i_{jk} v^j v^k \frac{\partial}{\partial v^i}, \quad (27)$$

where the Christoffel symbols Γ^i_{jk} are

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left[\frac{\partial g_{kl}}{\partial q^j} + \frac{\partial g_{jl}}{\partial q^k} - \frac{\partial g_{jk}}{\partial q^l} \right] \quad (28)$$

with g^{ij} being the inverse matrix of g_{ij} .

In the particular case we are considering where $g(v, w) = n^2 g_0(v, w)$, it was also shown above that the kernel of ω_L is two-dimensional and it is generated by $\Gamma_{\mathbb{L}}$ and the Liouville vector field Δ . The distribution defined by $\ker \omega_L$ is integrable because ω_L is closed; actually $[\Delta, \Gamma_{\mathbb{L}}] = \Gamma_{\mathbb{L}}$ and the distribution is also generated by Δ and K defined by $K = \frac{1}{v^3} \Gamma_{\mathbb{L}}$, for which $[\Delta, K] = 0$. In cartesian coordinates the Christoffel symbols are expressed as follows:

$$\Gamma^i_{jk} = \frac{1}{n} \left[\frac{\partial n}{\partial x^j} \delta_k^i + \frac{\partial n}{\partial x^k} \delta_j^i - \frac{\partial n}{\partial x^i} \delta_k^j \right]. \quad (29)$$

and the vector field K is given as

$$K = \frac{1}{v^3} \left[v^i \frac{\partial}{\partial x^i} - \left(\frac{2}{n} v^i (v \cdot \nabla n) - \frac{\|v\|^2}{n} \frac{\partial n}{\partial x^i} \right) \frac{\partial}{\partial v^i} \right]. \quad (30)$$

The theory of distributions suggests us the introduction of new local coordinates $y^i = F^i(x, v)$, $i = 1, \dots, 6$, adapted to the distribution defined by $\ker \omega_L$, i.e., such that $K = \frac{\partial}{\partial y^6}$, $\Delta = \frac{\partial}{\partial y^6}$ (see [12]). The search for these new coordinates is based on the solution of the partial differential equation system

$$KF^1 = 1, \quad \Delta F^1 = 0, \quad KF^2 = 0, \quad \Delta F^2 = 1,$$

and

$$KF^{2+a} = 0, \quad \Delta F^{2+a} = 0, \text{ for } a = 1 \dots, 4.$$

The explicit computation of these functions depends very much on the choice of the function $n(x^1, x^2, x^3)$. We will illustrate next the theory with an particular example. If n only depends on x^3 , the presymplectic form can be written in the way

$$\omega_L = dx^1 \wedge d\left(\frac{nv^1}{\sqrt{v^{12} + v^{22} + v^{32}}}\right) + dx^2 \wedge d\left(\frac{nv^2}{\sqrt{v^{12} + v^{22} + v^{32}}}\right) \quad (31)$$

$$+ \frac{nv^{32}}{(v^{12} + v^{22} + v^{32})^{3/2}} \left[v^1 d\left(\frac{v^1}{v^3}\right) \wedge dx^3 + v^2 d\left(\frac{v^2}{v^3}\right) \wedge dx^3 \right], \quad (32)$$

and the dynamical vector field is

$$\Gamma_{\mathbb{L}} = v^i \frac{\partial}{\partial x^i} - \frac{2}{n} v^1 v^3 \frac{dn}{dx^3} \frac{\partial}{\partial v^1} - \frac{2}{n} v^2 v^3 \frac{dn}{dx^3} \frac{\partial}{\partial v^2} \quad (33)$$

$$+ \frac{1}{n} (v^{12} + v^{22} - v^{32}) \frac{dn}{dx^3} \frac{\partial}{\partial v^3}. \quad (34)$$

After some calculations we find the solution of the former systems (see [3]). According to this, we will do the following choice for the new coordinates:

$$y^1 = x^1 - \frac{v^1}{v^2} x^2, \quad (35)$$

$$y^2 = x^2 - \int_0^{x^3} \frac{C_3}{\sqrt{(n^2(\zeta) - C_3^2)(1 + C_1^2)}} d\zeta, \quad (36)$$

$$y^3 = x^3 \quad (37)$$

$$y^4 = \frac{nv^1}{\sqrt{v^{12} + v^{22} + v^{32}}}, \quad (38)$$

$$y^5 = \frac{nv^2}{\sqrt{v^{12} + v^{22} + v^{32}}}, \quad (39)$$

$$y^6 = \log \left[n \sqrt{v^{12} + v^{22} + v^{32}} \right], \quad (40)$$

where

$$C_1 = \frac{v^1}{v^2} \quad (41)$$

and

$$C_3 = n \sqrt{\frac{v^{12} + v^{22}}{v^{12} + v^{22} + v^{32}}}, \quad (42)$$

doing the inverse change and after some easy calculations remains

$$\tilde{\omega}_L = d\left(y^1 + \frac{y^4}{y^5} y^2\right) \wedge dy^4 + dy^2 \wedge dy^5, \quad (43)$$

which shows that

$$\xi^1 = y^1 + \frac{y^4}{y^5} y^2 = x^1 - \frac{v^1}{v^2} \int_0^{x^3} \frac{C_3}{\sqrt{(n^2(\zeta) - C_3^2)(1 + C_1^2)}} d\zeta, \quad (44)$$

$$\xi^2 = x^2 - \int_0^{x^3} \frac{C_3}{\sqrt{(n^2(\zeta) - C_3^2)(1 + C_1^2)}} d\zeta \quad (45)$$

and the corresponding

$$\eta^1 = y^4 = \frac{nv^1}{\sqrt{v^{1^2} + v^{2^2} + v^{3^2}}}, \quad \eta^2 = y^5 = \frac{nv^2}{\sqrt{v^{1^2} + v^{2^2} + v^{3^2}}} \quad (46)$$

are Darboux coordinates for the symplectic form induced in the quotient space.

Let us now consider the particular but important case in which the refractive index becomes constant out of a region. If for $x^3 > L$, the index n is constant, the above mentioned Darboux coordinates ξ^1 and ξ^2 are

$$\xi^1 = x^1 - \frac{v^1}{v^2} \frac{C_3 x^3}{\sqrt{(n^2 - C_3^2)(1 + C_1^2)}}, \quad \xi^2 = x^2 - \frac{C_3 x^3}{\sqrt{(n^2 - C_3^2)(1 + C_1^2)}},$$

up to a constant, and from the expressions of C_1 and C_3 we see that the Darboux coordinates become

$$x^1 - \frac{v^1}{v^3} x^3, \quad x^2 - \frac{v^2}{v^3} x^3, \quad \frac{nv^1}{\sqrt{v^{1^2} + v^{2^2} + v^{3^2}}}, \quad \frac{nv^2}{\sqrt{v^{1^2} + v^{2^2} + v^{3^2}}}, \quad (47)$$

in full agreement with [13]. Therefore, for an optical system such that the refractive index depends only on x^3 and, furthermore, the region in which the index is not constant is bounded, we can choose Darboux coordinates by fixing a x^3 outside this region and taking Darboux coordinates for the corresponding problem of constant index. This justifies the choice of coordinates as usually done for the ingoing and outgoing light rays in the corresponding constant index media, i.e. it shows the convenience of using flat screens in far enough regions on the left and right respectively, and then this change of Darboux coordinates seems to be, from an active viewpoint, a canonical transformation (see [14]).

We will next find the symplectic structure arising in an anisotropic optical medium, as well as some Darboux coordinates for it. We recall what we are only considering anisotropic media for which the refractive index depends only on the ray direction, i.e., $n = n(v)$ but $\Delta n = 0$. In this case the presymplectic form remains as

$$\omega_L = \left[\|v\| \frac{\partial^2 n}{\partial x^j \partial v_i} + \frac{v_i}{\|v\|} \frac{\partial n}{\partial x^j} \right] dx^i \wedge dx^j \quad (48)$$

$$+ \left[\|v\| \frac{\partial^2 n}{\partial v_j \partial v_i} + \frac{v_i}{\|v\|} \frac{\partial n}{\partial v_j} + \frac{v_j}{\|v\|} \frac{\partial n}{\partial v_i} - \frac{n}{\|v\|^3} v_i v_j + \frac{n}{\|v\|} \delta_j^i \right] dx^i \wedge dv_j, \quad (49)$$

and the vector field associated with \mathbb{L}

$$\Gamma_{\mathbb{L}} = v_i \frac{\partial}{\partial x^i}. \quad (50)$$

We still have that

$$[\Delta, \Gamma_{\mathbb{L}}] = \Gamma_{\mathbb{L}}, \quad (51)$$

and then the $\ker \omega_L$ defines an involutive, and hence an integrable, distribution that is also generated by Δ and K , K being the vector field

$$K = \frac{1}{v_z} \Gamma_{\mathbb{L}} \quad (52)$$

commuting with Δ . In this way, we can find new local coordinates adapted to the distribution that allow us to find later on the symplectic form in the quotient manifold, by solving the following differential equation systems:

$$\Delta f = 0 \quad Kf = 0. \quad (53)$$

$$\Delta f = 1 \quad Kf = 0. \quad (54)$$

$$\Delta f = 0 \quad Kf = 1. \quad (55)$$

According to the solution of the former systems we will do the following choice for the new coordinates:

$$x_1 = \frac{v_x}{v_z}z - x, \quad x_2 = \frac{v_y}{v_z}z - y, \quad x_3 = z \quad (56)$$

$$y_1 = \frac{v_x}{v_z}, \quad y_2 = \frac{v_y}{v_z}, \quad y_3 = \log v_z, \quad (57)$$

the inverse change being given by

$$x = y_1 x_3 - x_1, \quad y = y_2 x_3 - x_2, \quad z = x_3 \quad (58)$$

$$v_x = y_1 \exp y_3, \quad v_y = y_2 \exp y_3, \quad v_z = \exp y_3. \quad (59)$$

Moreover, in the reduction of the presymplectic form we must point out that the condition $\Delta n = 0$ is written in the new coordinates as $\frac{\partial n}{\partial y_3} = 0$. Finally, using the change of coordinate former, we get after some calculations the following symplectic form in the quotient manifold

$$\tilde{\omega}_L = d \left[\frac{ny_1}{\sqrt{y_1^2 + y_2^2 + 1}} + \sqrt{y_1^2 + y_2^2 + 1} \frac{\partial n}{\partial y_1} \right] \wedge dx_1 \quad (60)$$

$$+ d \left[\frac{ny_2}{\sqrt{y_1^2 + y_2^2 + 1}} + \sqrt{y_1^2 + y_2^2 + 1} \frac{\partial n}{\partial y_2} \right] \wedge dx_2 \quad (61)$$

which is in full agreement with [13].

As a final comment, let us remark that, even in this case, if we restrict ourselves to a region of constant index, we recover the Darboux coordinates for constant index medium and we can think of the relation between the ingoing and outgoing light rays as a change of Darboux coordinates. Therefore this change of Darboux coordinates seems to be again, from an active viewpoint, a canonical transformation. The transformations of phase space will be, in general, non lineal, i.e., they generate optical aberrations. It is possible, finally, to analyse these aberrations using both group theoretical and Lie algebraic tools (see [10], [13], and references therein).

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